NBSIR 78-873

HIGHER ORDER MODES IN RECTANGULAR COAXIAL LINE WITH INFINITELY THIN INNER CONDUCTOR

John C. Tippet
David C. Chang

Department of Electrical Engineering
University of Colorado

March 1978

NBSIR 78-873

HIGHER ORDER MODES IN RECTANGULAR COAXIAL LINE WITH INFINITELY THIN INNER CONDUCTOR

John C. Tippet
David C. Chang

Department of Electrical Engineering
University of Colorado

CONTENTS

iii

HIGHER ORDER MODES IN RECTANGULAR COAXIAL LINE
WITH INFINITELY THIN INNER CONDUCTOR

JOHN C. TIPPET, Student Member, IEEE, and

DAVID C. CHANG, Senior Member, IEEE

ABSTRACT

The singular integral equation approach is used to derive
the secular equations for both TE and TM waves in a rectangular
coaxial line with zero thickness inner conductor. Approximations
for the secular equations are found that reduce to simple expres-
sions in terms of well-known special functions (elliptic integrals).
When the strip width is exceedingly small or nearly equal to the
width of the outer conductor, closed form expressions for the cut-
off frequencies can be found by replacing the elliptic integrals
by their asymptotic forms for modulus either near zero or one.

Key words: Higher order modes; rectangular coaxial line; striplines.

I. INTRODUCTION

In many applications involving rectangular coaxial waveguiding structures

an understanding of the propagation characteristics of higher order modes is

just as important as that of the fundamental TEM mode. One such application

is that of the so-called "TEM cell" which is used in some EMI measurement

systems [1]. This device consists of a section of rectangular coaxial line

which is used as a transducer for coupling EM energy from a device under

test into the TEM mode of the transmission line. The useful frequency range

of the TEM cell is limited by the cut-off frequencies of the higher order

modes. Thus, in order to design a cell for use over a specified frequency

range, these cut-off frequencies must be known.

The device analyzed in this paper consists of a zero thickness metal plate located inside a rectangular waveguide. The inner conductor may be offset vertically from its central position but is located symmetrically about the y-axis. Both the inner and outer conductor are perfectly conducting and the medium separating the two conductors is a homogeneous dielectric. A cross-sectional view of the rectangular line is shown in Fig. 1.

Although the transmission line properties of the TEM mode in these structures have been extensively studied, comparitively little work has been done in analyzing the higher order mode structure. One can find in the literature, however, this problem [2-4] as well as related problems, such as shielded strip lines [5-7], various coupling configurations [8-9], rectangular lines with thick inner conductors [10-13], and ridge waveguides [14-16] analyzed using a variety of techniques. These include functional equation techniques, finite difference techniques, mode matching techniques, integral equation techniques, and methods based on transmission line theory.

The purpose of this paper is to obtain a relatively simple, closed form expression for the characteristic equation for both TE and TM modes which is valid for arbitrary strip widths. Thus, very little computer programming is necessary to calculate the cut-off frequencies. In addition, approximate solutions for either small or large strip widths can be obtained without resorting to any numerical analysis. This is achieved by using the singular integral equation technique similar to that used in waveguide diaphragm problems [17]. This method has the advantage of handling the edge condition exactly and eliminates the problems encountered in any numerical solution associated with the discontinuities of the fields near the sharp edges of the inner conductor. More specifically, the problem is formulated using an integral equation--Green's function type of approach.

2

The singular part of the kernel of the resulting integral equation is extracted (as is done, e.g., in [18-20]) and the nonsingular part of the kernel that remains is expanding in terms of Chebyshev polynomials as suggested by [19].

II. FORMULATION

The cut-off frequencies of the higher-order modes of a rectangular coaxial line are just the eigenvalues K of the reduced wave equation

$$(\nabla^2 + K^2) \begin{pmatrix} H_z \\ E_z \end{pmatrix} = 0 \qquad (1)$$

where

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$$

and the unknowns satisfy the following boundary conditions on the metal walls

$$\partial_n H_z = 0$$

and

$$E_z = 0.$$

In the first case, H_z represents the z-component of the magnetic field of a TE mode; and in the second case, E_z represents the z-component of the electric field of a TM mode.

In order to solve this eigenvalue problem, we will convert (1) into an integral equation. This is accomplished by expanding $\begin{pmatrix} H_z \\ E_z \end{pmatrix}$ into a complete set of basis functions appropriate to each of the subregions above and below the strip (j=1,2, respectively), as

$$\begin{pmatrix} H_z^{(j)} \\ E_z^{(j)} \end{pmatrix} = \sum_{m,n} \begin{pmatrix} \overline{A}_{mn}^{(j)} & 0 \\ 0 & \overline{B}_{mn}^{(j)} \end{pmatrix} \begin{pmatrix} \psi_{mn}^{(j)} \\ \phi_{mn}^{(j)} \end{pmatrix} \qquad (j = 1,2)$$

3

where the $\bar{A}_{mn}^{(j)}$'s and $\bar{B}_{mn}^{(j)}$'s are as yet undeterminded coefficients.

For our geometry, $\psi_{mn}^{(j)}$ and $\phi_{mn}^{(j)}$ are just the TE and TM basis functions of a rectangular waveguide of height b_j, i.e.,

$$\psi_{mn}^{(j)}(x,y) = \left(\frac{2}{ab_j}\right)^{\frac{1}{2}}\cos\left(\frac{m\pi}{2a}(x+a)\right)\cos\left(\frac{n\pi y}{b_j}\right)$$

and

$$\phi_{mn}^{(j)}(x,y) = \left(\frac{2}{ab_j}\right)^{\frac{1}{2}}\sin\left(\frac{m\pi}{2a}(x+a)\right)\sin\left(\frac{n\pi y}{b_j}\right) \quad .$$

These basis functions also satisfy (1) but with known eigenvalues

$$K_{mn}^{(j)} = \left(\left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{b_j}\right)^2\right)^{\frac{1}{2}} \quad .$$

They do not, except in the rather trivial case of an unperturbed rectangular waveguide mode, satisfy the same continuity conditions as the unknowns $\begin{pmatrix} H_z \\ E_z \end{pmatrix}$ in the gap regions. We can still expand $\begin{pmatrix} H_z \\ E_z \end{pmatrix}$ in terms of these basis functions, however, since $\partial_n H_z$ or E_z need not be uniformly convergent at $y = 0$. In this case the limit $y \to 0$ cannot be taken inside the appropriate expansion. See [21] for a discussion of this phenomenon.

Using the orthogonality properties of the basis functions to solve for the unknown expansion coefficients, applying Green's theorem, and matching boundary conditions in the gap regions, results in the following integral equations for the TE and TM modes, respectively.

$$\int_w^a E_x(x',0)G^{TE}(x,x')dx' = 0 \tag{2}$$

and

$$P\int_w^a \partial_{x'}E_z(x',0)G^{TM}(x,x')dx' = 0 \quad . \tag{3}$$

4

where P denotes that the integral is to be interpreted in the principal value sense, $E_x(x,y) \propto \partial_y H_z(x,y)$, and the kernels G^{TE} and G^{TM} which contain the unknown eigenvalues K_{TE} and K_{TM} are given by

$$G^{TE}(x,x') = \sum_{j=1}^{2} \sum_{m,n} \Delta_m \Delta_n \frac{\psi_{mn}^{(j)}(x,0)\psi_{mn}^{(j)}(x',0)}{[K_{TE}^2 - K_{mn}^{(j)2}]} \tag{4}$$

and

$$G^{TM}(x,x') = \sum_{j=1}^{2} \frac{\partial}{\partial y} \left\{ \sum_{m,n} \frac{K_{on}^{(j)}}{K_{mo}^{(j)}} \frac{\phi_{mn}^{(j)}(x,y)\psi_{mn}^{(j)}(x',0)}{[K_{TM}^2 - K_{mn}^{(j)2}]} \right\} \Bigg|_{y=0} \tag{5}$$

where

$$\Delta_i = \begin{cases} \frac{1}{2} & i = 0 \\ 1 & i > 0 \end{cases} .$$

The symmetry about the y-axis allows the integrations appearing in (2) and (3) to be written only over the one gap from x = w to a and permits a decomposition of the TE and TM polarizations into even and odd modes. This is manifest in the summations on "m" in (4) and (5). For TE modes with odd symmetry, i.e., $H_z(x,y) = -H_z(-x,y)$, (4) is summed over odd values, while for TM modes with odd symmetry, i.e., $E_z(x,y) = -E_z(-x,y)$, (5) is summed over even values. The reverse is true for modes with even symmetry.

III.   EXTRACTION OF THE SINGULAR PARTS OF THE KERNEL FUNCTIONS

The summations on "n" in (4) and (5) can both be summed exactly using the residue series technique [22]. The derivative in (5) can then be taken and when evaluated at y = 0 one finds

$$G^{TE}(x,x') = \sum_{j=1}^{2} \frac{b_j^2}{2\pi} \sum_m \frac{\Delta m \cot \pi\gamma_m^{(j)}}{\gamma_m^{(j)}} \psi_{mn}^{(j)}(x,0)\psi_{mn}^{(j)}(x',0) \qquad (6)$$

and

$$G^{TM}(x,x') = -\frac{\pi}{2} \sum_{j=1}^{2} \sum_m \frac{\Gamma_m^{(j)} \cot \pi\Gamma_m^{(j)}}{K_{mo}^{(j)2}} \partial_x \psi_{mn}^{(j)}(x,0)\psi_{mn}^{(j)}(x',0) \quad (7)$$

where

$$\gamma_m^{(j)} = \frac{b_j}{\pi} [K_{TE}^2 - K_{mo}^{(j)2}]^{\frac{1}{2}} \qquad (8)$$

and

$$\Gamma_m^{(j)} = \frac{b_j}{\pi} [K_{TM}^2 - K_{mo}^{(j)2}]^{\frac{1}{2}} . \qquad (9)$$

Although the summations on "m" in (6) and (7) cannot be done in closed form, we can extract the singular part in each sum by replacing the coefficients in front of the basis functions by their asymptotic form for large "m", i.e.,

$$\frac{\cot \pi\gamma_m^{(j)}}{\gamma_m^{(j)}} \sim -\left(\frac{2a}{b_j}\right) \frac{1}{m}$$

and

$$\Gamma_m^{(j)} \cot \pi\Gamma_m^{(j)} \sim \left(\frac{b_j}{2a}\right) m .$$

The kernels will then consist of two parts, a singular part $\tilde{G}$ which can be summed in closed form (see, e.g., [22]) and a correction series $\hat{G} = G - \tilde{G}$. Recognizing that the summations on "m" are either over even or odd indices, four cases need to be considered. These are not independent, however, since it is easily verified that

6

$$\tilde{G}^{TM}(x,x') = \partial_x \tilde{G}^{TE}(x,x') \quad . \tag{10}$$

Thus, only the kernels for the TE polarization are needed, i.e.,

$$\tilde{G}^{TE}_{odd} = \frac{1}{\pi} \ln\left[\tan\frac{\pi}{4a}|t+t'| \tan\frac{\pi}{4a}|t-t'|\right] \tag{11}$$

and

$$\tilde{G}^{TE}_{even} = \frac{1}{\pi} \ln\left[4 \sin\frac{\pi}{2a}|t+t'| \sin\frac{\pi}{2a}|t-t'|\right] \tag{12}$$

where the following change of variables was introduced

$$t = a-x \tag{13}$$

and

$$t' = a-x'. \tag{14}$$

The singular kernels for the TM polarization can then be found using (10).

Although the correction series $\hat{G}$ cannot be summed in closed form, they are rapidly convergent and are given as

$$\hat{G}^{TE}(t,t') = \sum_m A_m \cos\left(\frac{m\pi t}{2a}\right)\cos\left(\frac{m\pi t'}{2a}\right) \tag{15}$$

and

$$\hat{G}^{TM}(t,t') = \sum_m B_m \sin\left(\frac{m\pi t}{2a}\right)\cos\left(\frac{m\pi t'}{2a}\right) \tag{16}$$

where

$$A_m = \sum_{j=1}^{2} \frac{b_j}{a\pi}\left(\frac{\Delta m \cot \pi\gamma_m^{(j)}}{\gamma_m^{(j)}} + \left(\frac{2a}{b_j}\right)\frac{(1-\delta_{mo})}{m}\right) \tag{17}$$

$$B_m = \sum_{j=1}^{2} \frac{-2}{mb_j}\left(\Gamma_m^{(j)}\cot\pi\Gamma_m^{(j)} - \left(\frac{b_j}{2a}\right)m\right) \tag{18}$$

7

and δ_{mo} is the Kroneker delta defined as

$$\delta_{mo} = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases} \quad .$$

IV. SOLUTION OF THE SINGULAR INTEGRAL EQUATIONS

Having extracted the singular parts of both kernels we can move the nonsingular correction series to the right-hand sides of both integral equations and treat these terms as forcing terms. Since the singular parts of the kernels of (2) and (3) are related according to (10), it is convenient to differentiate (2) with respect to x. Both equations will then be of the same canonical form. In terms of the new variables defined in (13) and (14), (2) and (3) can be rewritten as follows

$$P \int_0^g U(t')\tilde{G}^{TM}(t,t')dt' = \int_0^g U(t')\partial_t\hat{G}^{TE}(t,t')dt' \tag{19}$$

and

$$P \int_0^g V(t')\tilde{G}^{TM}(t,t')dt' = -\int_0^g V(t')\hat{G}^{TM}(t,t')dt' \tag{20}$$

where

$$U(t') \equiv E_x(x',0)$$

and

$$V(t') \equiv \partial_{x'}E_z(x',0) \quad .$$

Equations (19) and (20) are both singular integral equations. The standard form of the singular integral equation, however, has integration limits -1 and +1 respectively. These limits can be achieved through the use of the Schwinger transformation [17]

$$\cos\left(\frac{\pi t'}{a}\right) = \alpha + \beta u \qquad (21)$$

and

$$\cos\left(\frac{\pi t}{a}\right) = \alpha + \beta v \qquad (22)$$

where

$$\alpha = \frac{1}{2}\left[\cos\left(\frac{\pi g}{a}\right) + 1\right]$$

and

$$\beta = \frac{1}{2}\left[\cos\left(\frac{\pi g}{a}\right) - 1\right] \quad .$$

Defining

$$T_v[F(u)] \equiv \frac{1}{\pi}P\int_{-1}^{1}\frac{F(u)du}{u-v}$$

and transforming to the new variables u and v defined in (21) and (22) we find that (19) and (20) can be rewritten in standard form as

$$T_v[F_{ij}(u)] = H_{ij}(v) \qquad (i,j = 1,2) \qquad (23)$$

where

$$F_{11}(u) = U(t')\cos\left(\frac{\pi t'}{2a}\right)/\sin\left(\frac{\pi t'}{a}\right)$$

$$F_{12}(u) = U(t')/\sin\left(\frac{\pi t'}{a}\right)$$

$$F_{21}(u) = V(t')\cos\left(\frac{\pi t'}{2a}\right)/\sin\left(\frac{\pi t'}{a}\right)$$

$$F_{22}(u) = V(t')/\sin\left(\frac{\pi t'}{a}\right)$$

$$H_{11}(v) = \int_{0}^{g}U(t')\partial_t\hat{G}^{TE}(t,t')dt'/2\sin\left(\frac{\pi t}{2a}\right)$$

$$H_{12}(v) = \int_{0}^{g}U(t')\partial_t\hat{G}^{TE}(t,t')dt'/\sin\left(\frac{\pi t}{a}\right)$$

$$H_{21}(v) = -\int_{0}^{g}V(t')\hat{G}^{TM}(t,t')dt'/2\sin\left(\frac{\pi t}{2a}\right)$$

and

$$H_{22}(v) = -\int_{0}^{g}V(t')\hat{G}^{TM}(t,t')dt'/\sin\left(\frac{\pi t}{a}\right) \quad .$$

9

In (23) the subscript "i" refers to the type of mode, TE or TM respectively, and the subscript "j" refers to the type of symmetry, odd or even respectively.

As shown in [17] (23) can be inverted exactly for the unknown $F_{ij}(v)$ as

$$F_{ij}(v) = \frac{1}{\sqrt{1-v^2}} \{C_{ij} - T_v[\sqrt{1-u^2}\, H_{ij}(u)]\} \tag{24}$$

where the C_{ij}'s are constants to be determined.

In order to make use of (24) we need to express H_{ij} in terms of u. The most convenient expansion to use is one in terms of Chebyshev polynomials of the second kind U_n. In terms of these polynomials H_{ij} can be written as

$$H_{ij}(u) = \sum_{n=0}^{\infty} P_{ij,n}\, U_n(u) \tag{25}$$

where the $P_{ij,n}$'s are expansion coefficients to be determined.

To express F_{ij} as a function of v we can use the following identity [23]

$$T_v[\sqrt{1-u^2}\, U_n(u)] = -T_{n+1}(v) \tag{26}$$

where T_n is the Chebyshev polynomial of the first kind.

Thus, F_{ij} is given as

$$F_{ij}(v) = \frac{1}{\sqrt{1-v^2}} \{C_{ij} + \sum_{n=0}^{\infty} P_{ij,n} T_{n+1}(v)\} . \tag{27}$$

10

V. DERIVATION OF THE SECULAR EQUATIONS

In order to complete the solution given in (27) we need to evaluate the unknown constants C_{ij} and $P_{ij,n}$. We begin by first expressing C_{ij} in terms of the $P_{ij,n}$'s. For the TM polarization this is accomplished by requiring

$$\int_w^a \partial_{x'} E_z(x',0)dx' = E(a,0) - E(w,0) = 0. \tag{28}$$

For the TE polarization, we substitute the solution given in (27) back into the undifferentiated form of the integral equation, i.e., eq. (2). As shown in Appendix 1, one then finds for C_{ij} the following

$$C_{ij} = -\frac{1}{R_{ij,0}} \sum_{m=0}^{\infty} R_{ij,m+1} P_{ij,m} \tag{29}$$

where

$$R_{11,m} = I_{11,m} - \frac{\pi}{4\Delta_m} \sum_{n=m}^{\infty} A_{2n+1} P_{mn}$$

$$R_{12,m} = I_{12,m} + \frac{\pi}{2\Delta_m} \sum_{n=m}^{\infty} A_{2n} q_{mn}$$

$$R_{21,m} = I_{21,m}$$

and

$$R_{22,m} = 0 \quad (m > 0) \quad .$$

I_{11}, I_{12}, and I_{21} are three canonical integrals, the recursion formulae for which, are derived in Appendix II. For modes with odd symmetry, the integrals I_{11} and I_{12} can be expressed in terms of complete elliptic integrals of modulus $\sqrt{\alpha}$ for the TE polarization and $\sqrt{-\beta}$ for the TM polarization, while for modes with even symmetry, I_{12} contains no special functions.

11

p_{mn} and q_{mn} are the coefficients of the following expansions

$$\frac{\cos(2n+1)\left(\frac{\pi t'}{2a}\right)}{\cos\left(\frac{\pi t'}{2a}\right)} = \sum_{m=0}^{n} p_{mn} \, T_m(u)$$

and

$$\cos\left(\frac{n\pi t'}{a}\right) = \sum_{m=0}^{n} q_{mn} \, T_m(u) \quad .$$

We are now in a position to substitute our solution for F_{ij} (expressed solely in terms of the unknown $P_{ij,n}$'s), back into the definition of the H_{ij}'s defined in (23). Upon matching coefficients of $U_n(v)$ we then obtain an infinite set of equations for the unknown $P_{ij,n}$'s. Setting the determinant of the coefficients to zero we obtain the secular equation from which K can be determined. The details of this procedure are outlined in Appendix III. Four infinite sets of equations are derived of which the following for the odd TM polarization is representative.

$$P_{22,m} = \frac{a\beta}{2} \sum_{n=m}^{\infty} \sum_{k=0}^{n} B_{2n+2} \, s_{mn} \, q_{k+1,n+1} \, P_{22,k} \qquad (30)$$

where s_{mn} is the coefficient in the following expansion

$$\frac{\sin\left[(n+1)\frac{\pi t}{a}\right]}{\sin\left(\frac{\pi t}{a}\right)} = \sum_{m=0}^{n} s_{mn} \, U_m(v) \quad .$$

Equation (30) is the simplest of the four equations given in Appendix III since $C_{22} = 0$ for this case only. Fortunately, however, these equations can be greatly simplified. In most cases, only a few of the A_m's or B_m's, given respectively in (15) and (16), are significant, and thus the infinite matrix equations given in Appendix III reduce to equations of very small order. The number

of A_m's or B_m's that we must keep depends upon the number of higher order modes that we allow to propagate. We can see from (8) or (9) that once "m" becomes large enough so that $K_{mo} > K_{TE}$ or K_{TM}, respectively, then $\gamma_m^{(j)}$ or $\Gamma_m^{(j)}$ becomes imaginary. The cotangents then in (15) or (16) can be replaced by hyperbolic cotangents and thus the succeeding A_m's or B_m's become negligibly small. Thus, we can truncate our infinite matrix equation to a finite one at the sacrifice of being able to calculate all of the higher order modes. We will only find those modes for which K_{TE} or $K_{TM} \lesssim K_{Mo}$ where we have truncated the matrix to one of the finite order M.

In many applications (e.g., [24]) one is only interested in the first few higher order modes. This method is especially applicable for finding these modes since we can usually truncate the matrices to matrices of order one. In this case, (30) reduces to

$$1 = \frac{a\beta^2}{2} B_2 \tag{31}$$

since $s_{00} = 1$ and $q_{11} = \beta$. Substituting for B_2 from (16) we then find the following secular equation for the odd TM modes

$$\sum_{j=1}^{2} \frac{a\Gamma_2^{(j)}}{2b_j} \cot\pi\Gamma_2^{(j)} = 1 - \frac{1}{\beta^2} \tag{32}$$

where $\Gamma_2^{(j)}$ was defined in (8). Similarly, for the even TM modes we find

$$\sum_{j=1}^{2} \frac{a\Gamma_1^{(j)}}{b_j} \cot\pi\Gamma_1^{(j)} = \frac{2E-\alpha K}{2E-(1+\alpha)K} \qquad (\text{mod } \sqrt{-\beta}) \tag{33}$$

13

VI. CONCLUSIONS

In addition to the higher-order modes whose cut-off wavelengths are plotted in Figs. 2-4, there are also some higher-order modes which are unperturbed by the presence of the inner conductor. If $b_1 = b_2 = b$, these modes are just the $TE_{m,2n}$ and $TM_{m,2n}$ modes of the full guide of cross-section 2a x 2b. Alternatively, we can view this system as two rectangular sub-waveguides of cross-section 2a x b which are coupled through the gap. The sub-waveguide modes can then be combined to form a system mode of the entire structure which is either symmetric with respect to the inner conductor (i.e., $E_y(x,y) = E_y(x,-y)$ or antisymmetric. As found also in [27] only the symmetric combination is unperturbed by the presence of the inner conductor. If the inner conductor is offset then there exists fewer unperturbed modes.

Since the perturbed modes reduce to rectangular waveguide modes in the limit of either zero strip width ($\beta \rightarrow -1$) or zero gap ($\beta \rightarrow 0$) we can obtain approximate solutions for the cut-off wavelengths by replacing the cotangents and elliptic integrals in (32) to (34) by their asymptotic forms for beta either near 0 or -1. One can then find that the cut-off frequencies are perturbed as follows. For small strip widths

$$\delta^2_{TM_{11}} \propto 1/\ell n \left(\frac{8a}{\pi w}\right)$$

$$\delta^2_{TM_{21}} \propto \left(\frac{w}{a}\right)^2$$

and

$$\delta^2_{TE_{11}} \propto \left(\frac{w}{a}\right)^2$$

while for small gaps one finds

$$\delta^2_{TM_{11}} \propto \left(\frac{g}{a}\right)^2$$

$$\delta^2_{TM_{21}} \propto \left(\frac{g}{a}\right)^4$$

and

$$\delta^2_{TE_{11}} \propto 1/\ln\left(\frac{8a}{\pi g}\right)$$

where

$$K^2 - K^2_{unperturbed} \equiv \pm \left(\frac{\delta}{b}\right)^2 .$$

The significance of the logarithmic perturbation of the cut-off wavelength of the TE_{11} mode for small gap and the TM_{11} mode for small strip width is apparent in Figs. 3 and 4. For any finite gap width or strip width, respectively, the perturbation is quite significant. Thus, in contrast to the other cases for which the perturbation is algebraic, the cut-off frequencies are not accurately predicted by their unperturbed values. This same type of logarithmic or algebraic behavior was found by [8] in an analysis of a related problem of two rectangular waveguides coupled through a longitudinal slot.

In reference [11] are plotted similar curves for the case of a thick center conductor. The curves show the same qualitative behavior as a function of strip width. In particular, the cut-off wavelength of the TE_{11} mode for a small gap tends to be that of a "coaxial TE_{10} mode" in each sub-waveguide.

Although we have only presented the numerical results for a one-term approximation to the secular equations, higher-order solutions can be easily found, since the canonical integrals needed in the computation of the matrix elements are given recursively. For example, if we keep two of the B_m's in the expression for the secular equation for the odd TM mode, (30) reduces to

16

$$\left(1 - \frac{a\beta^2}{2} B_2\right) \left(1 - \frac{a\beta^4}{2} B_4\right) = 0 \quad . \tag{36}$$

The first term in (36) gives the same roots as we found earlier in our one-term approximation as given in (31). In addition, however, we have an extra term which gives rise to additional higher-order modes. One can see how continuing this process would ultimately recover all of the higher-order modes.

VII.   ACKNOWLEDGMENTS

VIII.   REFERENCES

[1]   Crawford, M. L., Generation of standard EM fields using TEM transmission
      cells, IEEE Trans. Electromagnetic Compatibility, vol. EMC-16, pp. 189-
      195 (Nov. 1974).

[2]   Belousova, L. I., and Sal'nikova, L. P., Computation of the cutoff
      frequencies of higher order modes in shielded symmetric stripline,
      Radiotekhnika (Khar'kov), No. 17, pp. 72-76 (1971).

[3]   Shestopalov, V. P., The Method of Riemann--Hilbert Problems in the
      Theory of Diffraction and Propagation of Electromagnetic Waves,
      Izdat. Khar'kov. University (1971).

[4]   Belousova, L. I., and Sal'nikova, L. P., Higher wave modes in a shielded
      nonsymmetrical strip line, Radio Eng. and Elect. Physics, Vol. 16,
      No. 7, pp. 1199-1202 (July 1971).

[5]   Lerer, A. M., and Mikhalevskii, V. S., Using paired integral equations in
      certain problems involving the diffraction of waveguide modes by bodies
      of finite size, Radiophys. Quantum Electron., vol. 18, pp. 859-865
      (1975).

[6]   Mittra, R., and Itoh, T., A new technique for the analysis of the disper-
      sion characteristics of microstrip lines, IEEE Trans. Microwave Theory
      Tech., vol. MTT-19, No. 1, pp. 47-56 (Jan. 1971).

[7]   Davies, J. B., and Corr, D. G., Computer analysis of the fundamental and
      higher-order modes in single and coupled microstrip, Electronics Letters,
      Vol. 6, No. 25, pp. 806-808 (Dec. 1970).

[8]   Garb, Kh. L., and Fridberg, I. Sh., Dispersional properties of a Rectangular
      waveguide, loaded in half-space by a narrow slot, in a wall of finite
      thickness, Radio Eng. and Elect. Physics. Vol. 19, No. 1 (1974).

18

[9] Voytovich, N. N., Katsenelenbaum, B. Z., and Sivov, A. N., The excitation of a two-dimensional metal cavity with a small opening (slotted cylinder), Radio Eng. and Elect. Physics, Vol. 19, No. 12, pp. 8-17 (Dec. 1974).

[10] Bezlyudova, M. M., Cut-off wavelength of waveguide-type oscillations in transmission lines having internal and external conductors of rectangular cross section, Radio Eng. and ELect. Physics, Vol. 8, No. 11, pp. 1727-1733 (Nov. 1963).

[11] W. Bräckelmann, D. Landmann and W. Schlosser, Die Grenzfrequenzen von Höheren Eigenwellen in Streifenleitungen, Arch Elek. Ubertragungstech., Vol. 21, No. 3, pp. 112-120 (Mar. 1967).

[12] Gruner, L., Higher order modes in rectangular coaxial waveguides, IEEE Trans. MTT, Vol. MTT-15, No. 8, pp. 483-485 (Aug. 1967).

[13] Bräckelmann, W., Wellentypen auf der Streifenleitung mit rechteckigem Schirm, Arch. Elek. Ubertragungstech., Vol. 21, No. 12, pp. 641-647 (1967).

[14] Pyle, J. R., The cutoff wavelength of the TE_{10} mode in ridged rectangular waveguide of any aspect ratio, IRE Trans. Microwave Theory Tech., Vol. MTT-14, No. 4, pp. 175-183 (April 1966).

[15] Collins, J. H., and Daly, P., Orthogonal mode theory of single ridge waveguide, J. Electronics and Control, vol. 17, pp. 121-129 (Aug. 1964).

[16] Cohn, S. B., Properties of ridge wave guide, Proc. IRE, pp. 783-788 (Aug. 1947).

[17] Lewin, L., Theory of Waveguides, London: Newnes-Butterworths (1975).

[18] Chang, D. C., Equivalent-circuit representation and characteristics of a radiating cylinder driven through a circumferential slot, IEEE Trans. Ant. Prop. Vol. AP-21, No. 6, pp. 792-796 (Nov. 1973).

[19] Gladwell, G. M. L., and Coen, S., A Chebyshev approximation method for microstrip problems, IEEE Trans. MTT, vol. MTT-23, No. 11, pp. 865-870, (Nov. 1975).

[20] Chang, W., and Otto, D. V., Current distributions on electrically thick tubular antennas of limited length, Int. J. Electronics, Vol. 39, No. 4, pp. 393-413 (Apr. 1975).

[21] Friedman, B., Principles and Techniques of Applied Mathematics, New York: Wiley, p. 271 (1956).

[22] Collin, R. E., Field Theory of Guided Waves, New York: McGraw-Hill (1960).

[23] Tricomi, F.G., Integral Equations, New York: Interscience (1957).

[24] Tippet, J. C., Chang, D. C., and Crawford, M. L., An analytical and experimental determination of the cutoff frequencies of higher-order TE modes in a TEM cell, National Bureau of Standards Internal Report NBSIR 76-841 (June 1976).

[25] Gradshteyn, I. S., and Ryzhik, I. M., Table of Integrals Series and Products, New York: Academic Press (1965).

[26] Byrd, P. F., and Friedman, M. D., Handbook of Elliptic Integrals for Engineers and Physicists, Berlin: Springer-Verlag (1954).

[27] Kopp, E. H., Coupling between dissimilar waveguides, IEEE Trans. on MTT, Vol. MTT-16, No. 1, pp. 6-11 (Jan. 1968).

APPENDIX 1. EXPRESSION FOR C_{ij} IN TERMS OF $P_{ij,n}$

We begin with the odd TM Polarization and transform (28) into an equation in terms of u as follows

$$\int_w^a \partial_{x'} E_z(x',0)dx' = \int_0^g V(t')dt' = \int_{-1}^1 F_{22}(u)du = 0 \qquad (1.1)$$

If we now substitute for $F_{22}(u)$ from (27), we obtain

$$C_{22} \int_{-1}^1 \frac{du}{\sqrt{1-u^2}} = - \sum_{n=0}^{\infty} P_{22,n} \int_{-1}^1 \frac{T_{n+1}(u)du}{\sqrt{1-u^2}} \quad . \qquad (1.2)$$

The right-hand side of (1.2) vanishes because of the orthogonality property of the Chebyshev polynomials, i.e., [25]

$$\int_{-1}^1 \frac{T_m(u) \ T_n(u)du}{\sqrt{1-u^2}} = \frac{\pi}{2\Delta_m} \delta_{mn} \qquad (1.3)$$

where

$$\Delta_m = \begin{cases} \frac{1}{2} & m=0 \\ 1 & m>0 \end{cases} \quad .$$

Thus,

$$C_{22} = 0 \ . \qquad (1.4)$$

For the even TM Polarization, we transform (1.1) into an equation in terms of a new variable χ defined as

$$\chi = \cos\left(\frac{\pi t'}{2a}\right) \qquad (1.5)$$

to obtain

21

$$\int_w^a \partial_{x'} E_z(x',0) dx' = 2 \int_0^g F_{21}(u) \sin\left(\frac{\pi t'}{2a}\right) dt' = \frac{4a}{\pi} \int_{\sqrt{\alpha}}^1 F_{21}\left(\frac{2\chi^2-1-\alpha}{\beta}\right) d\chi = 0.$$

If we now substitute for F_{21} from (27) we obtain

$$\int_{\sqrt{\alpha}}^1 \frac{C_{21} d\chi}{\sqrt{(1-\chi^2)(\chi^2-\alpha)}} = -\sum_{n=0}^{\infty} P_{21,n} \int_{\sqrt{\alpha}}^1 \frac{T_{n+1}\left(\frac{2\chi^2-1-\alpha}{\beta}\right) d\chi}{\sqrt{(1-\chi^2)(\chi^2-\alpha)}}$$

since

$$\sqrt{1-u^2} = -\frac{2}{\beta} \sqrt{(1-\chi^2)(\chi^2-\alpha)} \quad .$$

Defining $I_{21,n}$ as

$$I_{21,n} = \int_{\sqrt{\alpha}}^1 T_n\left(\frac{2\chi^2-1-\alpha}{\beta}\right) \frac{d\chi}{\sqrt{(1-\chi^2)(\chi^2-\alpha)}}$$

we find for C_{21} the following

$$C_{21} = -\frac{1}{I_{21,0}} \sum_{n=0}^{\infty} P_{21,n} I_{21,n+1} \tag{1.6}$$

Now for the odd TE polarization we transform (2) into an equation in terms of t and t', and without loss of generality, evaluate the resulting equation at t = 0 to obtain

$$\int_0^g U(t') \tilde{G}^{TE}(0,t') dt' = -\int_0^g U(t') \hat{G}^{TE}(0,t') dt' \quad . \tag{1.7}$$

The left-hand side (L.H.S.) of (1.7) can be transformed into an equation in terms of the variable χ defined in (1.5) as follows

$$\tilde{G}(0,t') = \frac{1}{\pi} \ln \tan^2 \frac{\pi t'}{4a} = \frac{1}{\pi} \ln\left(\frac{1-\chi}{1+\chi}\right) \quad .$$

22

Thus,

$$\text{L.H.S.} = \frac{2}{\pi} \int_0^g F_{11}(u) \ln\left(\frac{1-\chi}{1+\chi}\right) \sin\left(\frac{\pi t'}{2a}\right) dt' = \frac{4a}{\pi^2} \int_{\sqrt{\alpha}}^1 F_{11}\left(\frac{2\chi^2-1-\alpha}{\beta}\right) \ln\left(\frac{1-\chi}{1+\chi}\right) d\chi$$

If we now substitute for F_{11} from (27) we obtain

$$\text{L.H.S.} = \frac{2a}{\pi^2} \int_{\sqrt{\alpha}}^1 \frac{\ln\left(\frac{1+\chi}{1-\chi}\right) d\chi}{\sqrt{(1-\chi^2)(\chi^2-\alpha)}} \left\{ C_{11} + \sum_{n=0}^{\infty} P_{11,n} \, T_{n+1}\left(\frac{2\chi^2-1-\alpha}{\beta}\right) \right\} \cdot \qquad (1.8)$$

Defining $I_{11,n}$ as

$$I_{11,n} = \frac{1}{\pi} \int_{\sqrt{\alpha}}^1 T_n\left(\frac{2\chi^2-1-\alpha}{\beta}\right) \ln\left(\frac{1+\chi}{1-\chi}\right) \frac{d\chi}{\sqrt{(1-\chi^2)(\chi^2-\alpha)}}$$

we see that (1.8) reduces to

$$\text{L.H.S.} = \frac{2a\beta}{\pi} \left[C_{11} I_{11,0} + \sum_{n=0}^{\infty} P_{11,n} I_{11,n+1} \right] \cdot \qquad (1.9)$$

Now the right-hand side (R.H.S.) of (1.7) can be written in terms of u as follows

$$\text{R.H.S.} = -\int_0^g F_{11}(u) \frac{\sin\left(\frac{\pi t'}{a}\right)}{\cos\left(\frac{\pi t'}{2a}\right)} \sum_{n=0}^{\infty} A_{2n+1} \cos[(2n+1)\frac{\pi t'}{2a}] dt' =$$

$$= \frac{a\beta}{\pi} \sum_{n=0}^{\infty} A_{2n+1} \int_{-1}^1 F_{11}(u) \frac{\cos[(2n+1)\frac{\pi t'}{2a}]}{\cos\left(\frac{\pi t'}{2a}\right)} du.$$

If we now make use of the following expansion in terms of Chebyshev polynomials of the first kind T_m

23

$$\frac{\cos\left[(2n+1)\frac{\pi t'}{2a}\right]}{\cos\left(\frac{\pi t'}{2a}\right)} = \sum_{m=0}^{n} P_{mn} T_m(u) \tag{1.10}$$

and substitute for $F_{11}(u)$ from (27) we find the following

$$\text{R.H.S.} = \frac{a\beta}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_{2n+1} P_{mn} \int_{-1}^{1} \frac{T_m(u)\,du}{\sqrt{1-u^2}} \left[C_{11} + \sum_{k=0}^{\infty} P_{11,k} T_{k+1}(u)\right]. \tag{1.11}$$

The integrations in (1.11) can be evaluated by using the orthogonality relation given in (1.3) and thus (1.11) reduces to

$$\text{R.H.S.} = a\beta \sum_{n=0}^{\infty} A_{2n+1} \left[P_{on} C_{11} + \frac{(1-\delta_{no})}{2} \sum_{k=1}^{n} P_{11,k-1} P_{kn}\right] \tag{1.12}$$

where δ_{no} is the Kroneker delta. Finally (1.9) and (1.12) when equated and solved for C_{11} gives

$$C_{11} = \frac{-\left(\sum_{n=0}^{\infty} P_{11,n} I_{11,n+1} - \frac{\pi}{4} \sum_{n=0}^{\infty} A_{2n+1}(1-\delta_{no}) \sum_{k=1}^{n} P_{11,k-1} P_{kn}\right)}{\left(I_{11,0} - \frac{\pi}{2} \sum_{n=0}^{\infty} A_{2n+1} P_{on}\right)}. \tag{1.13}$$

For the even TE polarization we again start with (1.7), but we transform the left-hand side of that equation into an equation in terms of u as follows

$$\tilde{G}^{TE}(0,t') = \frac{1}{\pi} \ln\left[(-2\beta)(1+u)\right]$$

Thus,

$$\text{L.H.S.} = -\frac{a\beta}{\pi^2} \int_{-1}^{1} F_{12}(u) \ln\left[(-2\beta)(1+u)\right] du .$$

24

If we now substitute for $F_{12}(u)$ from (27) we obtain

$$\text{L.H.S.} = -\frac{a\beta}{\pi^2} \int_{-1}^{1} \frac{\ln[(-2\beta)(1+u)]du}{\sqrt{1-u^2}} \left[C_{12} + \sum_{n=0}^{\infty} P_{12,n} T_{n+1}(u)\right]. \quad (1.14)$$

Defining $I_{12,n}$ as

$$I_{12,n} = \frac{1}{\pi} \int_{-1}^{1} \frac{T_n(u)\ln(1+u)}{\sqrt{1-u^2}} du + \delta_{no} \ln(-2\beta)$$

we see that (1.14) reduces to

$$\text{L.H.S.} = -\frac{a\beta}{\pi} \left[C_{12} I_{12,0} + \sum_{n=0}^{\infty} P_{12,n} I_{12,n+1}\right]. \quad (1.15)$$

Now the right-hand side of (1.7) can be written in terms of u as follows

$$\text{R.H.S.} = -\int_{0}^{g} F_{12}(u) \sin\left(\frac{\pi t'}{a}\right) \sum_{n=0}^{\infty} A_{2n} \cos\left(\frac{n\pi t'}{a}\right) dt' =$$

$$= \frac{a\beta}{\pi} \sum_{n=0}^{\infty} A_{2n} \int_{-1}^{1} F_{12}(u) \cos\left(\frac{n\pi t'}{a}\right) du.$$

If we now make use of the following expansion

$$\cos\left(\frac{n\pi t'}{a}\right) = \sum_{m=0}^{n} q_{mn} T_m(u)$$

and substitute for $F_{12}(u)$ from (27) we find the following

$$\text{R.H.S.} = \frac{a\beta}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_{2n} q_{mn} \int_{-1}^{1} \frac{T_m(u)du}{\sqrt{1-u^2}} \left[C_{12} + \sum_{k=0}^{\infty} P_{12,k} T_{k+1}(u)\right]. \quad (1.16)$$

25

Again, the integrations in (1.16) can be evaluated using the orthogonality relation given in (1.3) to obtain

$$\text{R.H.S.} = a\beta \sum_{n=0}^{\infty} A_{2n} \left[q_{on} C_{12} + \frac{(1-\delta_{no})}{2} \sum_{k=1}^{n} P_{12,k-1} q_{kn} \right] \qquad (1.17)$$

Finally (1.15) and (1.17) when equated and solved for C_{12} gives

$$C_{12} = - \frac{\left[\sum_{n=0}^{\infty} P_{12,n} I_{12,n+1} + \frac{\pi}{2} \sum_{n=0}^{\infty} A_{2n}(1-\delta_{no}) \sum_{k=1}^{n} P_{12,k-1} q_{kn} \right]}{\left(I_{12,0} + \pi \sum_{n=0}^{\infty} A_{2n} q_{on} \right)} \qquad .(1.18)$$

Equations (1.4), (1.6), (1.13) and (1.18) can all be expressed in one equation as

$$C_{ij} = - \frac{1}{R_{ij,0}} \sum_{m=0}^{\infty} R_{ij,m+1} P_{ij,m}$$

where

$$R_{11,m} = I_{11,m} - \frac{\pi}{4\Delta_m} \sum_{n=m}^{\infty} A_{2n+1} P_{mn}$$

$$R_{12,m} = I_{12,m} + \frac{\pi}{2\Delta_m} \sum_{n=m}^{\infty} A_{2n} q_{mn}$$

$$R_{21,m} = I_{21,m}$$

and

$$R_{22,m} = 0 \quad (m > 0) \qquad .$$

26

APPENDIX 2. EVALUATION OF THE CANONICAL INTEGRALS

The canonical integrals appearing in (29) are defined by

$$I_{11,n} = \frac{1}{\pi} \int_{\sqrt{\alpha}}^{1} T_n\left(\frac{2\chi^2-1-\alpha}{B}\right) \ell n\left(\frac{1+\chi}{1-\chi}\right) \frac{d\chi}{\sqrt{(1-\chi^2)(\chi^2-\alpha)}}$$

$$I_{12,n} = \frac{1}{\pi} \int_{-1}^{1} \frac{T_n(u)\,\ell n(1+u)}{\sqrt{1-u^2}}\ du + \delta_{no}\ell n(-2\beta)$$

and

$$I_{21,n} = \int_{\sqrt{\alpha}}^{1} T_n\left(\frac{2\chi^2-1-\alpha}{\beta}\right) \frac{d\chi}{\sqrt{(1-\chi^2)(\chi^2-\alpha)}}$$

The evaluation of I_{12} can be found in [19] and is given by

$$I_{12,n} = \begin{cases} \ell n(-\beta) & n = 0 \\[2mm] \dfrac{(-1)^{n+1}}{n} & n > 0 \end{cases}$$

The remaining two integrals can both be evaluated by recognizing that [25]

$$T_n(u)-T_{n+2}(u) = 2(1-u^2)U_n(u)$$

where

$$\sqrt{1-u^2} = \frac{-2}{\beta}\sqrt{(1-\chi^2)(\chi^2-\alpha)}$$

and

$$u = \frac{2\chi^2-1-\alpha}{\beta}$$

27

Thus, if we take $I_{i1,n} - I_{i1,n+2}$ we can convert the integral into one for which the square root is now in the numerator. This integral can then be integrated by parts to obtain the following recursion relation

$$I_{i1,n+2} = \frac{2\left(\frac{1+\alpha}{1-\alpha}\right)(2n+2)\,I_{i1,n+1} - (2n+1)\,I_{i1,n} + J_{i1,n}}{(2n+3)} \qquad (i = 1,2) \quad (2.1)$$

where

$$J_{11,n} = \frac{16}{\pi\beta^2}\int_{\sqrt{\alpha}}^{1} U_n\left(\frac{2\chi^2-1-\alpha}{\beta}\right)\left(\frac{\chi^2-\alpha}{1-\chi^2}\right)^{\frac{1}{2}}\chi\;d\chi$$

and

$$J_{21,n} = 0\;.$$

J_{21} is zero because the integrand of I_{21} does not contain a logarithmic function as does I_{11}. The evaluation of J_{11} is easily accomplished by substituting

$$\chi^2 = \cos^2\theta + \alpha\sin^2\theta$$

to give

$$J_{11,n} = \frac{4}{\beta}\,(-1)^{n+1}\;.$$

The starting values for the recurrence relations given in (2.1) are given as [26]

$$I_{11,0} = K\,(\sqrt{\alpha})$$

$$I_{11,1} = \frac{1}{\beta}\left\{2\left[1+K(\sqrt{\alpha})-E(\sqrt{\alpha})\right] - (1+\alpha)\,K(\sqrt{\alpha})\right\}$$

$$I_{21,0} = K(\sqrt{-\beta})$$

28

and

$$I_{21,1} = \frac{1}{\beta} \left[2E(\sqrt{-\beta}) - (1+\alpha) \; K \; (\sqrt{-\beta}) \right]$$

where K and E are complete elliptic integrals of the first and second kind, respectively.

APPENDIX 3. DERIVATION OF THE INFINITE MATRIX EQUATIONS

The procedure for deriving the infinite matrix equations for the four cases, even and odd TE and TM, is basically the same. Therefore, we will derive only the odd TE case and quote the results for the remaining cases. We begin with the definition at H_{11} defined in (23), i.e.,

$$H_{11}(v) = \int_0^g U(t') \partial_t \hat{G}^{TE}(t,t') \, dt' \, / 2\sin(\tfrac{\pi t}{2a}) \tag{3.1}$$

From the definition of $\hat{G}^{TE}$ given in (15) we can derive the following result

$$\frac{\partial_t \hat{G}^{TE}}{2\sin(\tfrac{\pi t}{2a})} = \frac{\pi}{4a} \sum_{n=0}^{\infty} (2n+1)A_{2n+1} \cos\left(\tfrac{\pi t'}{2a}\right) \left\{ \frac{\sin[(2n+1)\tfrac{\pi t}{2a}]}{\sin(\tfrac{\pi t}{2a})} \quad \frac{\cos[(2n+1)\tfrac{\pi t'}{2a}]}{\cos(\tfrac{\pi t'}{2a})} \right\}. \tag{3.2}$$

If we use the following expansion in (3.2),

$$\frac{\sin[(2n+1)\tfrac{\pi t}{2a}]}{\sin(\tfrac{\pi t}{2a})} = \sum_{m=0}^{n} r_{mn} U_m(v)$$

insert (3.2) into (3.1), and use the expression for H_{11} given in (25) we obtain the following

$$\sum_{n=0}^{\infty} P_{11,n} U_n(v) = \frac{\pi}{4a} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (2n+1)A_{2n+1} r_{mn} \int_0^g U(t')\cos\left(\tfrac{\pi t'}{2a}\right) \left\{ \frac{\cos[(2n+1)\tfrac{\pi t'}{2a}]}{\cos\tfrac{\pi t'}{2a}} \right\} dt' U_m(v).$$

$$\tag{3.3}$$

We can now equate coefficients of $U_n(v)$ to obtain

$$P_{11,m} = \frac{\pi}{4a} \sum_{n=m}^{\infty} (2n+1)A_{2n+1} r_{mn} \int_0^g V(t') \cos\left(\tfrac{\pi t'}{2a}\right) \left\{ \frac{\cos[(2n+1)\tfrac{\pi t'}{2a}]}{\cos(\tfrac{\pi t'}{2a})} \right\} dt' \tag{3.4}$$

30

If we now transform the integration in (3.4) from t' to u via (21) and use the expansion given in (1.10), we obtain the following

$$P_{11,m} = \frac{\beta}{4} \sum_{n=m}^{\infty} \sum_{k=0}^{n} (2n+1) A_{2n+1} \, r_{mn} \, P_{kn} \int_{-1}^{1} F_{11}(u) \, T_k(u) \, du \; . \qquad (3.5)$$

Next, we insert the expression for $F_{11}(u)$ given in (27) and note that the integration in (3.5) can be performed using the orthogonality relation given in (1.3). Equation (3.5) then reduces to the following

$$P_{11,m} = \frac{\pi\beta}{8} \sum_{n=m}^{\infty} \sum_{k=0}^{n} \frac{(2n+1) \, A_{2n+1} \, r_{mn} \, P_{kn} \, P_{11,k-1}}{\Delta_k}$$

where we have defined $P_{11,-1} \equiv C_{11}$.

The final step involves substituting for C_{11} from (29) with the result that

$$P_{11,m} = \frac{\pi\beta}{8} \sum_{n=m}^{\infty} (2n+1) \, A_{2n+1} \, r_{mn} \left\{ (1-\delta_{no}) \sum_{k=0}^{n-1} P_{k+1,n} \, P_{11,k} - \frac{2p_{on}}{R_{11,0}} \sum_{k=0}^{\infty} R_{11,k+1} \, P_{11,k} \right\} .$$
$$(3.6)$$

Equation (3.6) is thus the infinite matrix equation desired. The following equations for the other three cases can be derived in an analogous manner.

$$P_{12,m} = \frac{\pi\beta}{2} \sum_{n=m}^{\infty} (n+1) \, A_{2n+2} \, s_{mn} \left\{ \sum_{k=0}^{n} q_{k+1,n+1} \, P_{12,k} - \frac{2q_{o,n+1}}{R_{12,0}} \sum_{k=0}^{\infty} R_{12,k+1} \, P_{12,k} \right\}$$

$$P_{21,m} = \frac{a\beta}{4} \sum_{n=m}^{\infty} B_{2n+1} \, r_{mn} \left\{ (1-\delta_{no}) \sum_{k=0}^{n-1} P_{k+1,n} \, P_{21,k} - \frac{2p_{on}}{R_{21,0}} \sum_{k=0}^{\infty} R_{21,k+1} \, P_{21,k} \right\}$$

31

and

$$P_{22,m} = \frac{a\beta}{2} \sum_{n=m}^{\infty} \sum_{k=0}^{n} B_{2n+2} \, s_{mn} \, q_{k+1,n+1} \, P_{22,k}$$

where we have made use of the following expansion

$$\frac{\sin\left(\frac{(n+1)\pi t}{a}\right)}{\sin\left(\frac{\pi t}{a}\right)} = \sum_{m=0}^{n} s_{mn} \, U_m(v) \quad .$$
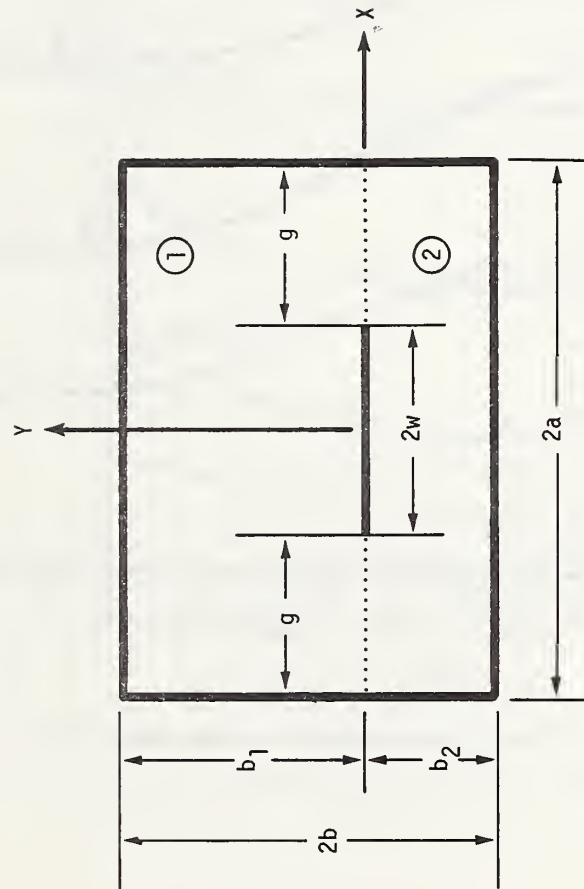
32

Fig. 1. Cross Section of a rectangular coaxial line.
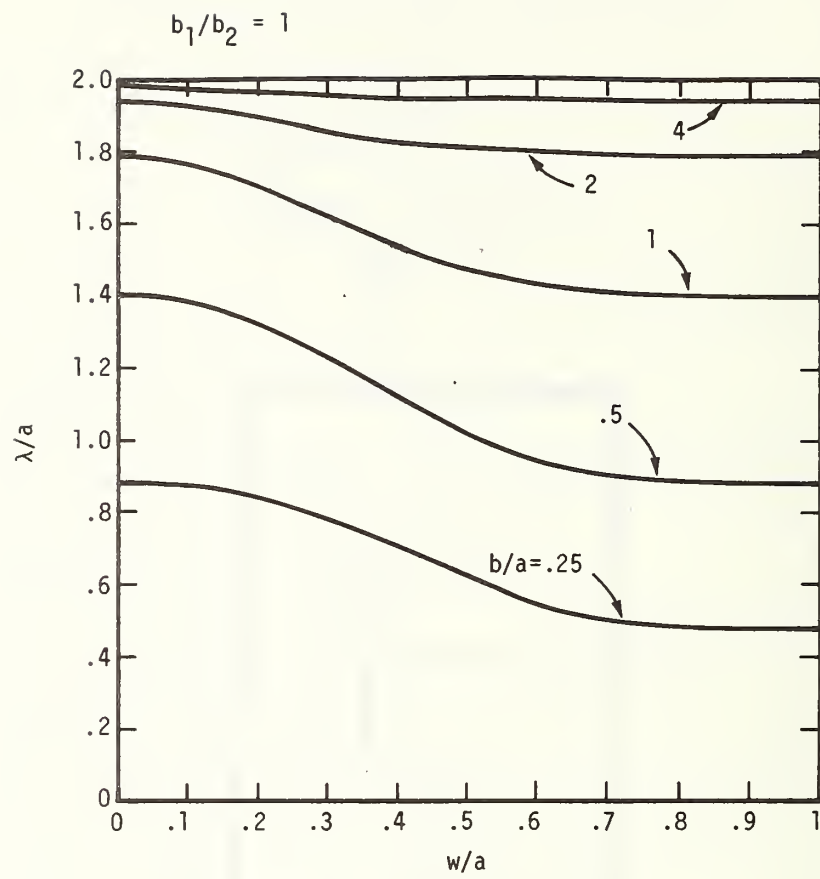
33

$b_1/b_2 = 1$

λ/a

w/a

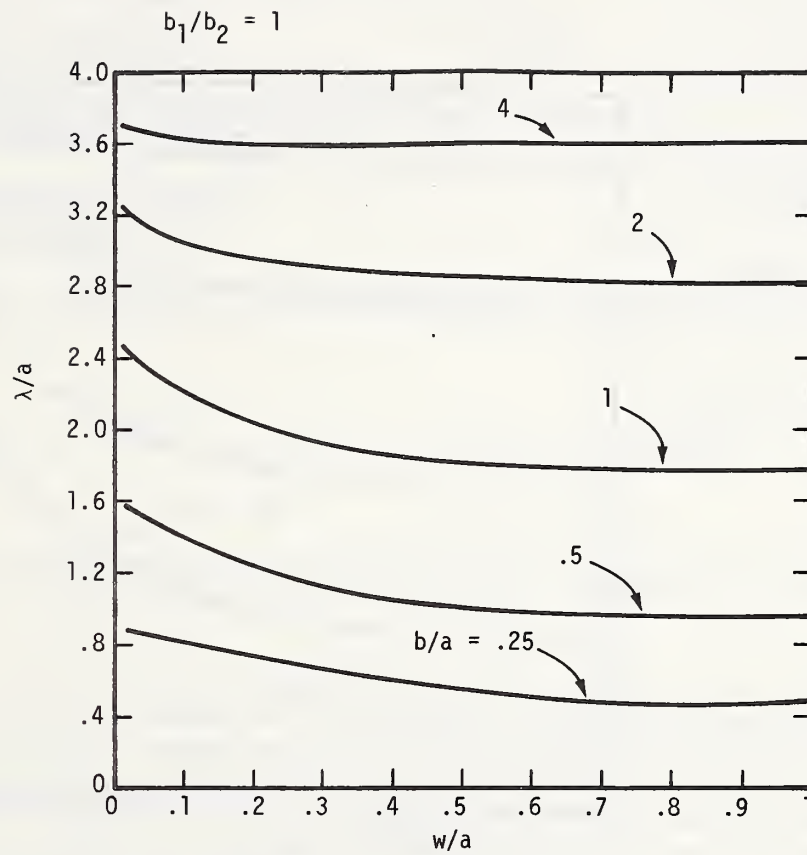Fig. 2. Cutoff wavelength vs. strip width of the TM_{21} mode.

34

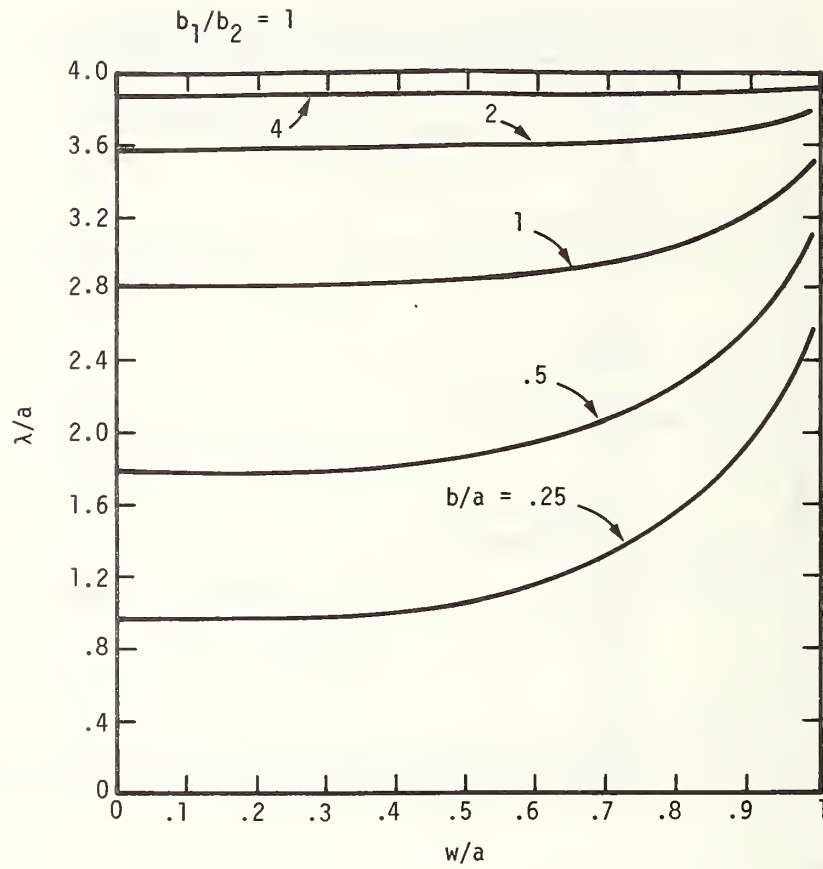Fig. 3. Cutoff wavelength vs. strip width of the TM_{11} mode.

Fig. 4. Cutoff wavelength vs. strip width of the TE_{11} mode.

U.S. DEPT. OF COMM. BIBLIOGRAPHIC DATA SHEET	1. PUBLICATION OR REPORT NO. NBSIR 78-873	2. Gov't Accession No.	3. Recipient's Accession No.

4. TITLE AND SUBTITLE Higher Order Modes in Rectangular Coaxial Line with Infinitely Thin Inner Conductor	5. Publication Date March 1978
	6. Performing Organization Code 276.00

7. AUTHOR(S) John C. Tippet and David C. Chang	8. Performing Organ. Report No.

9. PERFORMING ORGANIZATION NAME AND ADDRESS NATIONAL BUREAU OF STANDARDS DEPARTMENT OF COMMERCE WASHINGTON, D.C. 20234	10. Project/Task/Work Unit No. 2763280
	11. Contract/Grant No. NOAACST-8393

12. Sponsoring Organization Name and Complete Address (Street, City, State, ZIP) Same as #9.	13. Type of Report & Period Covered
	14. Sponsoring Agency Code

15. SUPPLEMENTARY NOTES

16. ABSTRACT (A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)

The singular integral equation approach is used to derive the secular equations for both TE and TM waves in a rectangular coaxial line with zero thickness inner conductor. Approximations for the secular equations are found that reduce to simple expressions in terms of well-known special functions (elliptic integrals). When the strip width is exceedingly small or nearly equal to the width of the outer conductor, closed form expressions for the cut-off frequencies can be found by replacing the elliptic integrals by their asymptotic forms for modulus either near zero or one.

17. KEY WORDS (six to twelve entries; alphabetical order; capitalize only the first letter of the first key word unless a proper name; separated by semicolons)

Higher order modes; rectangular coaxial line; striplines.

18. AVAILABILITY [X] Unlimited	19. SECURITY CLASS (THIS REPORT)	21. NO. OF PAGES
[ ] For Official Distribution. Do Not Release to NTIS	UNCLASSIFIED	42
[ ] Order From Sup. of Doc., U.S. Government Printing Office Washington, D.C. 20402, SD Cat. No. C13	20. SECURITY CLASS (THIS PAGE)	22. Price
[X] Order From National Technical Information Service (NTIS) Springfield, Virginia 22151	UNCLASSIFIED	$4.50